GENERALISED MAGNUS MODULES OVER THE BRAID GROUP

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Abstract. W. Magnus' representations of submonoids $E \leq \mathrm{End}(F)$ of the endomorphisms of a free group F of finite rank are generalised by identifying them with the first homology group of F with particular coefficient modules. By considering a suitable free resolution of the integers over the semidirect product of free groups, a class of representations of the braid group can be obtained on higher homology groups. The resolution shows that the holonomy representations of the braid group and of the Hecke algebra constructed topologically by R. J. Lawrence belong to this class.

Contents

1. Introduction

These notes were prepared for seminar talks. They are based on some of the ideas sketched in the author's thesis [10]. It is shown that the class of Magnus representations of the braid group, cf.[2], can be suitably generalised via group homology to include the topological representations of R. J. Lawrence [8].

W. Magnus, in order to give a derivation for the well-known Burau representation of the braid group, described a class of representations of automorphism groups of free groups, see e.g. [11]. J. Birman devoted a chapter of her book to this class of "Magnus representations", cf.[2]. Among these representations are e.g. the representation found by W. Burau, which leads to a construction of the Alexander polynomial via a Markov functional, as well as those constructed by B. Gassner. G. D. Mostow implicitly used Magnus representations of the unpermuting braid group to investigate the monodromy group of Euler-Picard integrals [13]. This is just to mention some of the applications.

The Magnus representation modules (in the sense originally considered by W. Magnus) can be understood as the first homology groups of the free group with particular coefficient modules. Using Artin's imbedding $B_n \to \mathrm{Aut}(F_n)$, a semidirect product $B_n \ltimes F_n$ of the braid group B_n with the free group F_n of rank n can be defined. F_n is imbedded as a normal subgroup into this product. By functoriality of H_* the

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quotient group $(B_n \ltimes F_n)/F_n \simeq B_n$ acts onto $H_1(F_n, M)$, if M is a module over $B_n \ltimes F_n$. We will show that this construction generalises the approaches of Magnus and Birman.

These considerations show that Magnus' representations are the first nontrivial examples of a general procedure: one may use any pair $K \lhd G$ of groups with quotient $G/K \simeq B_n$ and any G module N to obtain a B_n module $H_*(K, N)$.

A step in this direction has been undertaken by R. J. Lawrence, cf.[8], from a different and geometrical point of view. In an attempt to understand the Jones polynomial in geometrical terms, she constructs a vector bundle with a natural flat connection over a base space having the braid group B_n as its fundamental group. In this way she obtains braid representations from the holonomy of the connection, which under certain conditions factor through the Hecke algebra. The typical fiber of the bundle is the m-th homology $H_m(Y; \chi)$ of the configuration space Y of m distinct points in the n-fold punctured plane with a suitably chosen abelian local coefficient system $\chi \in \mathrm{Hom}(\pi_1(Y, y), \mathbb{C}\backslash\{0\})$. The class of representations obtained in this way e.g. can be used for the construction of the one-variable Jones polynomial [9]. For a short account on this approach, cf.[1]. The case $m = 1$ was investigated with different motivation in [13].

In the present note, it will be shown that, since Y is an Eilenberg-MacLane complex of type $(\pi_1(Y, y), 1)$, Lawrence's construction via Eilenberg's theorem on local coefficients is the topological approach to the construction of the homology of the group $\pi_1(Y, y)$ with coefficient module $(\chi, \mathbb{C})$. The fundamental group $\pi_1(Y, y)$ can be imbedded as a normal subgroup into a generalised braid group $B_{n,m}$ with quotient $B_{n,m}/\pi_1(Y, y) \simeq B_n$. Therefore Lawrence's approach fits into the algebraic setting sketched above. In particular $\pi_1(Y, y) \simeq F_n$ for $m = 1$ and we recover Magnus' representations.

We will therefore, after having reviewed Magnus' original construction, find a free $\pi_1(Y, y)$ resolution $\partial \in \mathrm{Hom}_{\pi_1(Y,y)}(C)$ of the integers. This resolution is supposed by the recursive structure of $\pi_1(Y, y)$. It explicitly allows the computation of the B_n action onto chain complexes $N \otimes_{\pi_1(Y,y)} C$ by means of braid ring valued matrices. A special version of these matrices was found topologically in [8]. In their general form they are the "braid valued Burau matrices" derived independently and by different means in [5], [10]. The name is due to the fact that the classical Burau matrices are images of the aforementioned ones under a ring homomorphism of the matrix elements. The braid valued matrices encode all the braid representations mentioned above in a simple and unified form.

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2. Magnus representations of submonoids of $\mathrm{End}(F)$

This section presents a description of the "classical" situation in a more general formulation. The approaches of [2] and [11] will be linked together.

In the following, E is a general submonoid of the monoid $\mathrm{End}(F)$ of right endomorphisms of the free group F. A subscript indicates the finite rank n of F, if necessary. Since the endomorphisms are acting from the right, the product ab is the endomorphism $b \circ a$, b after a. The semidirect product $E \ltimes F$ is the monoid consisting of the set $E \times F$ equipped with the product $(a, f)(b, g) = (ab, b(f)g)$, $a, b \in E$, $f, g \in F$ and with unity $(1, 1)$. We will identify E and F with their images in $E \ltimes F$ and therefore have $af = (a, 1)(1, f) = (a, f)$, $E \ltimes F = EF$ and $afbg = abb(f)g$. F

is normal in EF in the sense that for every $p \in EF$ and $f \in F$ there is a $g \in F$ uniquely determined by $fp = pg$. The linear extension of any group (or monoid) homomorphism $\phi \in \mathrm{Hom}(G, H)$ to the group ring $\mathbb{Z}G$ is the map ϕ_*. The relative augmentation ideal $I_{N,G} = \mathrm{Ker}(\phi_*) \lhd \mathbb{Z}G$ is generated by the subset $N - 1 \subset \mathbb{Z}G$, where N is the normal subgroup $N = \mathrm{Ker}(\phi) \lhd G$. We set $I_{G,G} = I_G$ if we consider full augmentation ideals.

1 (W. Magnus)**.** *Let $U \lhd F$ be a normal subgroup. Let $E = St_{\mathrm{End}(F)}(F/U) \leq \mathrm{End}(F)$ be the stabilizing monoid of F/U in $\mathrm{End}(F)$. Then $U/[U,U]$ is an F/U-E bimodule: $[f] \in F/U$ maps $[u] \in U/[U,U]$ to $[fuf^{-1}]$, $a \in E$ maps $[u]$ to $[a(u)]$, and the actions of E and F/U commute.*

This statement is taken from [11], p.471. The assignments defined by $[f]$ and a are mappings: let $u, p \in U$, $c \in [U,U]$. Then $fpucp^{-1}f^{-1} = fupp^{-1}u^{-1}pucp^{-1}f^{-1} = fuf^{-1}c'$ for some $c' \in [U,U]$. Therefore the image of $[u]$ under $[f]$ is uniquely determined. For $a \in E$ we have $a(uc) = a(u)a(c)$ and $a(c) \in [U,U]$, such that a too defines a map. Since E acts trivially onto F/U, $a(f) = fp$ for some $p \in U$. Thus, $a(fuf^{-1}) = fpa(u)p^{-1}f^{-1} = fa(u)f^{-1}fp[p^{-1}, a(u)^{-1}]p^{-1}f^{-1}$, where $fp[p^{-1}, a(u)^{-1}]p^{-1}f^{-1} \in [U,U]$, and the operations of E and F/U commute. $\Diamond$

Using the Fox derivation, Birman has given another construction related to the Magnus modules $U/[U,U]$. For any epimorphism $\psi \in \mathrm{Hom}(F, H)$ let M_ψ be the free left $\mathbf{Z}H$ module of rank n with basis $\{s_1, \dots, s_n\}$. Let $\delta_\psi \in \mathrm{Der}(F, M_\psi)$ be the total Fox derivation, cf. [2], pp.104. It is a group derivation uniquely defined as a map by the equations

$$\begin{aligned} \delta_\psi(f_i) &:= s_i, \\ \delta_\psi(fg) &:= \delta_\psi(f) + \psi(f)\delta_\psi(g), \end{aligned}$$

for a free generating system $\{f_1, \dots, f_n\}$ of F and $f, g \in F$. δ_ψ has an extension $\bar{\delta}_\psi \in \mathrm{Der}(\mathbb{Z}F, M_\psi)$ to a ring derivation, defined as the unique homomorphism of abelian groups with $\bar{\delta}_\psi(f) = \delta_\psi(f)$, $f \in F$. We drop the bar distinguishing both derivations. Let $\partial_{j,\psi} \in \mathrm{Der}(F, \mathbb{Z}H)$ be the partial Fox derivations with respect to the generators f_j, $\delta_\psi(f) = \sum_j \partial_{j,\psi}(f)s_j$, $f \in F$. When we consider the case $\psi = \mathrm{Id}$ we drop indices referring to ψ.

2 (J. Birman)**.** *Let $E \leq \mathrm{End}(F)$ be the maximal monoid satisfying $\psi \circ a = \psi$ for all $a \in E$. For every $a \in E$ let $\rho(a)$ be the endomorphism of M_ψ uniquely defined by*

$$s_i \mapsto \delta_\psi(a(f_i)).$$

Then $\rho \in \mathrm{Hom}(E, \mathrm{End}(M_\psi))$.

Statement and proof follow [2], thm.3.9, p.116. For $a \in E$ let the "Fox-Jacobi matrix" with respect to the generators $\{f_i\}$ be defined by $a_{i,j} = \partial_j(a(f_i))$. Due to the chain rule for the Fox derivation, [2], prop.3.3, pp.105, for $a, b \in E$ holds,

$$\begin{aligned} (ab)_{i,j} &= \partial_j(b(a(f_i))) \\ &= \sum_k b(\partial_k(a(f_i)))\partial_j(b(f_k)) \\ &= \sum_k b(a_{i,k})b_{k,j}. \end{aligned}$$

Using $\rho(a)(s_i) = \delta_\psi(a(f_i)) = \sum_j \partial_{j,\psi}(a(f_i))s_j = \sum_j \psi_*(a_{i,j})s_j$, we find,

$$\begin{aligned}\rho(ab)(s_i) &= \sum_j \psi_*((ab)_{i,j})s_j \\ &= \sum_{j,k} \psi_*(b(a_{i,k})b_{k,j})s_j \\ &= \sum_{j,k} \psi_*(a_{i,k})\psi_*(b_{k,j})s_j \\ &= \rho(b)(\rho(a)(s_i)).\end{aligned}$$

Thus, $\rho \in \mathrm{Hom}(E, \mathrm{End}(M_\psi))$. $\diamondsuit$

In order to understand the relationship between the Magnus modules $U/[U,U]$ and the Birman modules M_ψ, let $\mu_\psi \in \mathrm{Hom}(F, H \ltimes M_\psi)$ be the representation of F that is associated to the derivation $\delta_\psi \in \mathrm{Der}(F, M_\psi)$: the semidirect product of H and M_ψ is defined by $(g,p)(h,q) = (gh, p+gq)$ for $g,h \in H$ and $p,q \in M_\psi$ and μ_ψ maps $f \mapsto (\psi(f), \delta_\psi(f))$. Since δ_ψ is a group derivation, μ_ψ is a homomorphism (called the "Magnus ψ representation" of F in [2]). It is characterised by

3 (Blanchfield, Birman)**.**

$$\begin{aligned}\mathrm{Ker}(\mu_\psi) &= [\mathrm{Ker}(\psi), \mathrm{Ker}(\psi)], \\ \mathrm{Image}(\mu_\psi) &= \{(h, \sum h_j s_j); h \in H, h_j \in \mathbb{Z}H, h-1 = \sum h_j(\psi(f_j)-1)\}.\end{aligned}$$

By theorems of Blanchfield, cf. [2], thm.3.5, pp.107, and Magnus, cf. [2], thm.3.7, pp.111, the kernel is identified. The image is determined by [2], thm.3.6, pp.108. $\diamondsuit$

Both constructions are related by the following fact.

4. *Given either a Magnus module $U/[U,U]$ or a Birman module M_ψ, there is a short exact sequence of $\mathbb{Z}H$-E bimodules,*

$$0 \to U/[U,U] \to M_\psi \to Q \to 0,$$

with $H \simeq F/U$, $\psi \in \mathrm{Hom}(F,H)$ such that $\mathrm{Ker}(\psi) = U$, and $E = St_{\mathrm{End}(F)}(F/U)$.

Let $U \lhd F$ be given, $E = \mathrm{St}_{\mathrm{End}(F)}(F/U)$, giving rise to the Magnus module $U/[U,U]$ over F/U and E. Let $\psi \in \mathrm{Hom}(F, F/U)$ be the projection homomorphism. Then $\psi(a(f)) = \psi(fp) = \psi(f)$, with some $p \in U$, since E acts as identity onto F/U. So the projection ψ is E-invariant and it defines a Birman module M_ψ. We must find a monomorphism from the F/U-E bimodule $U/[U,U]$ into M_ψ. The Magnus representation of F, μ_ψ, induces an isomorphism of the group $F/[U,U]$ onto its image. The restriction of this monomorphism to $U/[U,U]$ satisfies $\mu_\psi([u]) = (1, \delta_\psi(u))$, for $u \in U = \mathrm{Ker}(\psi)$. So $\delta_\psi : U/[U,U] \to M_\psi$ has the desired properties: it is an injective homomorphism of abelian groups due to Magnus' theorem on $\mathrm{Ker}(\mu_\psi)$. It intertwines the operation of $\mathbb{Z}(F/U)$ onto $U/[U,U]$ with the left multiplication in M_ψ,

$$\begin{aligned}\delta_\psi(fuf^{-1}) &= \delta_\psi(f) + \psi(f)\delta_\psi(u) - \psi(fuf^{-1})\delta_\psi(f) \\ &= \psi(f)\cdot\delta_\psi(u).\end{aligned}$$

It also intertwines the right E actions on the two modules, $\delta_\psi(a([u])) = \rho(a)(\delta_\psi([u]))$. Indeed, from $u = f_{i_1}^{\epsilon_1}\dots f_{i_k}^{\epsilon_k}$, $\epsilon_j \in \{-1,1\}$ follows

$$\delta_\psi(u) = \sum_{j=1}^{k} \psi(f_{i_1}^{\epsilon_1}\dots f_{i_{j-1}}^{\epsilon_{j-1}})\epsilon_j\psi(f_{i_j})^{(\epsilon_j-1)/2}s_{i_j}$$

and

$$\delta_\psi(a(u)) = \sum_{j=1}^{k} \psi(a(f_{i_1}^{\epsilon_1}\dots f_{i_{j-1}}^{\epsilon_{j-1}}))\epsilon_j\psi(a(f_{i_j}^{(\epsilon_j-1)/2}))\delta_\psi(a(f_{i_j})).$$

Now the E-invariance of ψ can be used and by comparison of the expressions the claim is proved.

Conversely, let ψ be given and be E-invariant. $U = \mathrm{Ker}(\psi)$ is an E-invariant (since $\psi(a(u)) = \psi(u) = 1$ for $u \in U$) normal subgroup of F. E acts as identity onto F/U, since $f^{-1}a(f) \in U$, so $U/[U,U]$ is a Magnus module with respect to E. Again the Fox derivation $\delta_\psi \in \mathrm{Der}(U, M_\psi)$ yields a monomorphism $U/[U,U] \to M_\psi$. $\diamondsuit$

We call this sequence the "Birman-Magnus sequence". The image $\delta_\psi(U)$ of the Magnus module in the Birman module "measures" how far the set $\{\psi(f_j) - 1\}$ is from being linearly independent over $\mathbf{Z}H$.

5. Image$(\delta_\psi : \mathrm{Ker}(\psi) \to M_\psi) = \{\sum_j h_j s_j ; h_j \in \mathbf{Z}H, \sum h_j(\psi(f_j) - 1) = 0\}$.

Let $u \in \mathrm{Ker}(\psi)$ and $\delta_\psi(u) = \sum h_j s_j$, for some $h_j \in \mathbb{Z}H$. Then $u - 1 \in I_F$, where I_F is the augmentation ideal of F in the ring $\mathbb{Z}F$, such that $u - 1 = \sum u_j(f_j - 1)$ for suitable (and unique) $u_j \in \mathbb{Z}F$. We apply $\delta_\psi \in \mathrm{Der}(\mathbb{Z}F, M_\psi)$ to find $\sum h_j s_j = \sum \psi_*(u_j)s_j$ and $h_j = \psi_*(u_j)$. Then $\psi_*(u - 1) = 0$ implies, $\sum h_j(\psi(f_j) - 1) = 0$. Conversely, let $\sum h_j(\psi(f_j) - 1) = 0$, $h_j \in \mathbb{Z}H$. By Lyndon's theorem, [2], thm.3.6, pp.108, there is a $u \in \mathrm{Ker}(\psi)$ with $\delta_\psi(u) = \sum h_j s_j$. $\diamondsuit$

Now we will construct a short exact sequence of EF-E bimodules that generalises the Birman-Magnus sequence. Let $\eta \in \mathrm{Hom}(EF, E)$ be the projection, i.e. $\eta(af) = a$ for $a \in E$, $f \in F$. The extension $\eta_* \in \mathrm{Hom}(\mathbb{Z}(EF), \mathbb{Z}E)$ to the group ring defines the relative augmentation ideal $\mathrm{Ker}(\eta_*) = I_{F,EF}$ of F in $\mathbb{Z}EF$. We consider the ideal $I_{F,EF}$ as left EF right E bimodule. Since it is free as a left EF module, there is a representation of E in terms of $\mathbb{Z}(EF)$-valued matrices. In section 5 we will find these matrices for the case of the braid group.

6. *Let N be a right EF module. Then there is a short exact sequence of right E modules,* $0 \to H_1(F, N) \to N \otimes_{EF} I_{F,EF} \to NI_{F,EF} \to 0$.

We have a free resolution $0 \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon_*} \mathbb{Z} \to 0$ of the integers over the free group F, where $P_1 = I_F$ is the augmentation ideal, $P_0 = \mathbb{Z}F$ is the group ring and $\epsilon_* : \mathbb{Z}F \to \mathbb{Z}$ is the augmentation homomorphism, cf. [14], thm.11.3.2, p.322. By the exact sequence $0 \to H_1(F, N) \to N \otimes_F \partial_1(P_1) \to N \otimes_F P_0$, cf. [14], thm.11.2.7, p.319, we have $H_1(F, N) \simeq \mathrm{Ker}(N \otimes_F I_F \to N \otimes_F \mathbb{Z}F)$. Finally we notice that $N \otimes_F I_F \simeq N \otimes_{EF} I_{F,EF}$ (on the right hand side, the structure as right E module is more obvious) and that the maps $N \otimes_F I_F \to N \otimes_F \mathbb{Z}F$ and $N \otimes_{EF} I_{F,EF} \to NI_{F,EF}$ have isomorphic kernels, the later map is onto and the kernel is preserved by the right E action. Thus, we obtain the short exact sequence of right E modules as claimed. $\diamondsuit$

We will show that this sequence specialises to the Birman-Magnus sequence. Let $\psi \in \mathrm{Hom}(F, H)$ be an epimorphism with $\psi \circ a = \psi$ for every $a \in E$. This condition allows us to construct an extension $\bar\psi \in \mathrm{Hom}(EF, H)$, by setting $\bar\psi(af) = \psi(f)$, $a \in E$, $f \in F$. We regard $\mathbb{Z}H$ as a right EF module using $\bar\psi$, i.e. $af : h \mapsto h\bar\psi(af) = h\psi(f)$.

7.
1. *The Birman module M_ψ as left $\mathbb{Z}H$ right E bimodule is isomorphic to $\mathbb{Z}H \otimes_{EF} I_{F,EF}$.*
2. *Let $U = \mathrm{Ker}(\psi)$. The corresponding Magnus module $U/[U,U]$ is isomorphic to $H_1(F, \mathbb{Z}H)$.*

1) The restriction of δ_ψ to I_F is a homomorphism of left F modules, $\delta_\psi(f(f_i - 1)) = \delta_\psi(ff_i) - \delta_\psi(f) = \psi(f)\delta_\psi(f_i - 1)$. So there is a map

$$\mathbb{Z}H \otimes_{EF} I_{F,EF} \simeq \mathbb{Z}H \otimes_F I_F \xrightarrow{1\otimes\delta_\psi} \mathbb{Z}H \otimes_F M_\psi \simeq M_\psi.$$

We claim it is an isomorphism of $\mathbb{Z}H$-E bimodules. Since $\mathbb{Z}H \otimes_F I_F$ and $\mathbb{Z}H \otimes_F M_\psi$ are free left $\mathbb{Z}H$ modules with bases $\{1 \otimes (f_j - 1)\}$ and $\{1 \otimes s_j\}$, respectively, and

$(1 \otimes \delta_\psi)(1 \otimes (f_i - 1)) = 1 \otimes s_j$, the map is an isomorphism of left $\mathbb{Z}H$ modules. We now consider the right E action. In $\mathbb{Z}H \otimes_{EF} I_{F,EF}$ for $a \in E$ we have $1 \otimes (f_j - 1)a = 1 \otimes a(a(f_j) - 1)$ which is equal to $1 \otimes (a(f_j) - 1) \in \mathbb{Z}H \otimes_F I_F$ due to the condition $\bar{\psi}(a) = 1$. By $1 \otimes \delta_\psi$ the element $1 \otimes (a(f_j) - 1)$ is mapped to $1 \otimes \delta_\psi(a(f_j))$, such that indeed we have a homomorphism of right E modules.

2) The Birman-Magnus sequence together with theorem (5) identifies the Magnus module $U/[U, U]$ with the submodule $\{\sum_j h_j s_j; h_j \in \mathbf{Z}H, \sum h_j \psi_*(f_j - 1) = 0\} \leq M_\psi$ via δ_ψ. This submodule is isomorphic to $\mathrm{Ker}(\mathbb{Z}H \otimes_F I_F \to \mathbb{Z}H \otimes_F \mathbb{Z}F) \simeq H_1(F, \mathbb{Z}H)$ by the map $\sum h_j s_j \mapsto \sum h_j \otimes (f_j - 1)$. $\Diamond$

We thus have generalised the Birman-Magnus modules $\mathbb{Z}H \otimes_{EF} I_{F,EF}$ insofar, as we allow the condition $\psi \circ a = \psi$ to be dropped and any right EF module N be used instead of the ring $\mathbb{Z}H$, which is a particular right EF module.

There are no higher nontrivial homology groups than $H_1(F, N)$. But by choosing E as Artin's braid group B_n, one may use semidirect products of free groups in order to construct sequences of modules over B_n from higher homology groups. The homology module discussed so far is the first member of this sequence.

3. Generalities on the braid group

From now on we specialise our considerations from $E \leq \mathrm{End}(F)$ to Artin's braid group. In this section we collect some well known facts about it. Eventually we show that the topological homology groups appearing as the fibers in the geometrical approach to braid representations in [8] can be understood as group homology.

The braid group B_n on n strings (over the Euclidean plane) is the group generated by the $n-1$-set $\{\tau_i; i \in \{1, \ldots, n-1\}\}$ according to the relations of Artin, cf.[2], lemma 1.8.2, pp.20:

$$\begin{aligned} \tau_i \tau_j &= \tau_j \tau_i, \text{ if } \mid i - j \mid \geq 2, \\ \tau_i \tau_{1+i} \tau_i &= \tau_{1+i} \tau_i \tau_{1+i}. \end{aligned}$$

We set $\tau_{i,i} := \vartheta_{i,i} := 1$ and use the abbreviations $(i, j, k, l \in \{1, \ldots, n\}; i < j)$

$$\begin{aligned} \tau_{i,j} &:= \tau_i \tau_{1+i} \ldots \tau_{j-2} \tau_{j-1}, \\ \tau_{j,i} &:= \tau_{j-1} \tau_{j-2} \ldots \tau_{1+i} \tau_i, \\ \vartheta_{i,j} &:= \tau_{j-1,i}^{-} \tau_{j-1}^2 \tau_{j-1,i} =: \vartheta_{j,i}. \end{aligned}$$

8. *For $k \geq 2$ let B_k be generated by the set $\{\tau_i^{(k)}; 1 \leq i \leq k-1\}$ according to Artin's relations. The map $\tau_i^{(n)} \mapsto \tau_i^{(1+n)}$ uniquely extends to a monomorphism $B_n \to B_{1+n}$.*

The map extends to a homomorphism, since the relations of B_n are taken into relations of B_{1+n}. This homomorphism is injective, since all additional relations of B_{1+n} not being relations of B_n involve the generator τ_n, such that the $\tau_i^{(1+n)}$ for $i \leq n-1$ obey the same relations as do the generators of B_n. $\Diamond$

9. *Let $\{t_i; 1 \leq i \leq n-1\}$ be the set of permutations mapping $i \mapsto i+1$, $i+1 \mapsto i$, $\{i, i+1\} \not\ni j \mapsto j$. The mapping $\pi : \tau_i \mapsto t_i$ uniquely extends to an epimorphism $\pi \in \mathrm{Hom}(B_n, S_n)$ from the braid group onto the group S_n of permutations of the numbers $\{1, \ldots, n\}$ by mapping $\alpha\beta \mapsto \pi(\alpha)\pi(\beta)$.*

The t_i exactly satisfy $t_i^2 = 1$ and the Artin relations of the braid group. Furthermore they generate S_n: an n-permutation π can be achieved by first commuting the element $\pi^-(n)$ into its place by the t_i, leading to a problem in $n-1$ elements. $\Diamond$

10. *Let P_n be the kernel of $\pi \in \mathrm{Hom}(B_n, S_n)$, called the pure or non-permuting braid group. The elements*

$$\vartheta_{i,j} := \tau_i^- \tau_{1+i}^- \dots \tau_{j-2}^- \tau_{j-1}^2 \tau_{j-2} \dots \tau_{1+i} \tau_i =: \vartheta_{j,i},$$

$i < j$, $i, j \in \{1, \dots, n\}$ generate P_n. The relations between the generators $\vartheta_{i,j}$ are the consequences of

$$\vartheta_{k,l}^{-\epsilon} \vartheta_{i,j} \vartheta_{k,l}^{\epsilon} = \begin{cases} \vartheta_{i,j}, & i < k \text{ or } l < i \\ \mathrm{Ad}((\vartheta_{k,j}\vartheta_{l,j})^{\epsilon})(\vartheta_{i,j}), & i = k \text{ or } i = l \\ \mathrm{Ad}([\vartheta_{l,j}^{\epsilon}, \vartheta_{k,j}^{\epsilon}]^{-\epsilon})(\vartheta_{i,j}), & k < i < l \end{cases}, \tag{1}$$

$i < j$, $k < l < j$, $\epsilon \in \{-1, 1\}$, $\mathrm{Ad}(x)(y) := xyx^-$, $[x, y] := xyx^-y^-$, for $x, y \in P_n$.

A proof might proceed by finding a Schreier transversal (cross-section) to B_n/P_n in B_n. Cf.[2], pp.20, where two pairs of parentheses are missing in lines 10 and 11. A detailed algebraic proof is given in [6], pp.153. ♢

The next theorem by recursion defines a normal series for P_n.

11. *Let $U_n \le P_n \le B_n$ be the subgroup of the non-permuting braid group generated by the set $\{\vartheta_{n,i}; i \in \{1, \dots, n-1\}\}$. (This is the subgroup of the braid group B_n in which only the n-th string is not held fixed.) Then P_n is isomorphic to the semidirect product $P_n \simeq P_{n-1} \ltimes U_n$, where P_{n-1} is imbedded into P_n (by identifying the first $n-1$ strings): $\vartheta_{i,j}^{(n-1)} \mapsto \vartheta_{i,j}^{(n)}$. P_{n-1} acts onto U_n by conjugation.*

Statement and proof following [2], pp.22. By the relations eqn. (1) U_n is normal in P_n. The quotient P_n/U_n is isomorphic to the group generated by $\{\vartheta_{i,j}; 1 \le i < j \le n-1\}$, that is P_{n-1}. P_{n-1} is imbedded into P_n s.t. the exact sequence $1 \to U_n \to P_n \to P_{n-1} \to 1$ splits. So the result follows, cf.[14], p.304. ♢

Let $X_n := \{(x_1, \dots, x_n) \in \mathbb{C}^n;\ \text{if } i \ne j \text{ then } x_i \ne x_j\}$ be the configuration space of n ordered distinct points in the plane. For $x \in X_n$ let $Y_{x,m} := \{(y_1, \dots, y_m) \in X_m; y_i \ne x_j\}$ be the configuration space of m ordered distinct points in the plane with n points x_j removed.

12. *The projection $Y_{x,m+k} \to Y_{x,m}$ onto the first m components is a fiber bundle with the fiber over $y \in Y_{x,m}$ being $Y_{(x,y),k} = \{(z_1, \dots, z_k) \in X_k; z_i \ne x_j, z_i \ne y_l\}$.*

A theorem of Faddell and Neuwirth, cf.[2], thm.1.2, pp.12. Or cf.[6]. ♢

13. *The projection $X_n \to X_n/S_n$ under the symmetric group S_n is a regular $n!$ sheeted S_n covering.*

Cf.[2], prop.1.1, p.11. ♢

The following result on the fiber $Y_{x,m}$ is needed for the algebraical approach to homology groups.

14. *$Y_{x,m}$ is an Eilenberg-MacLane space of type $(\pi_1(Y_{x,m}), 1)$.*

For every k we choose $x \in X_k$ and set $Y_{k,m} = Y_{x,m}$. The fiber bundle $Y_{n+m,k} \to Y_{n,m+k} \to Y_{n,m}$ has the following exact homotopy sequence,

$$\dots \xrightarrow{\partial} \pi_i(Y_{n+m,k}) \xrightarrow{i_*} \pi_i(Y_{n,m+k}) \xrightarrow{p_* \circ j} \pi_i(Y_{n,m}) \xrightarrow{\partial} \pi_{i-1}(Y_{n+m,k}) \dots$$

By induction assumption, $\pi_i(Y_{n+m,k}) \simeq \{1\} \simeq \pi_i(Y_{n,m})$ for every $i > 1$, every $n \ge 0$ and some m and k. This is true for $k = m = 1$ at least, since in this case $Y_{n+m,k}$ and $Y_{n,m}$ are Euclidean planes with $n+m$ and n punctures, respectively. We conclude that also the homotopy groups $\pi_i(Y_{n,m+k})$ vanish for $i > 1$. ♢

We need further preparation in order to determine the structure of the fundamental group $\pi_1(Y_{x,m})$.

15. 1. *The normal subgroup $U_n \lhd P_n$ is free over the set $\{\vartheta_{n,i}; i \in \{1, \dots, n-1\}\}$.*

2. *The non-permuting braid group* P_n *is isomorphic to the fundamental group* $\pi_1(X_n)$.
3. *We have an isomorphism* $B_n \simeq \pi_1(X_n/S_n)$.

This topological proof follows [2], pp.22. For an algebraical proof of the first statement, cf.[6], pp.153.

1) From the homotopy sequence of the fiber bundle $Y_{n-1,1} \to X_n \to X_{n-1}$ we obtain the exact sequence

$$1 \xrightarrow{\partial} \pi_1(Y_{n-1,1}) \xrightarrow{i_*} \pi_1(X_n) \xrightarrow{p_* \circ j} \pi_1(X_{n-1}) \xrightarrow{\partial} 1. \tag{2}$$

By topological reasons we can find classes of paths $[\gamma(\vartheta_{i,j})] \in \pi_1(X_k)$ corresponding to the generators of P_k and obeying the pure braid relations, at least. So for $k \in \{n-1, n\}$ there are epimorphisms $h_k \in \mathrm{Hom}(P_k, \pi_1(X_k))$ (which, in fact, will be shown to be isomorphisms, once the present statement has been proved) sending $\vartheta_{i,j} \mapsto [\gamma(\vartheta_{i,j})]$. The restriction $h_n \mid_{U_n}$ maps a generator $\vartheta_{n,j}$ to a loop running once around the j-th puncture, starting and terminating at the basepoint of the punctured plane $Y_{n-1,1}$. Since $\pi_1(Y_{n-1,1})$ is a free group over these $n-1$ loops, we have surjections $\pi_1(Y_{n-1,1}) \to U_n \to \pi_1(Y_{n-1,1})$, where the first map sends a free generator, i.e. a loop around the j-th puncture, to $\vartheta_{n,j}$. Since a free group of finite rank is Hopfian, cf.[14], thm.6.1.12, p.159, this composed surjective endomorphism must be an automorphism and U_n must be isomorphic to $\pi_1(Y_{n-1,1})$.

2) The exact sequence $1 \to U_n \to P_n \to P_{n-1} \to 1$ together with the short exact sequence (2) of homotopy groups yields a commutative diagram where $U_n \to \pi_1(Y_{n-1,1})$ is an isomorphism. Assume, $P_{n-1} \to \pi_1(X_{n-1})$ for some n is an isomorphism as well. Then also $P_n \to \pi_1(X_n)$ must be such. Since $\pi_1(X_1) \simeq P_1$ are trivial groups, the statement is proved by induction.

3) This statement follows from the exact homotopy sequence $1 \to \pi_1(X_n) \to \pi_1(X_n/S_n) \to S_n \to 1$ of the regular covering $X_n \to X_n/S_n$. One uses the fact that there are isomorphisms $P_n \to \pi_1(X_n)$ and a commutative diagram of two short exact sequences. ♢

Therefore with the help of theorem (11) we may identify P_n with the iterated semidirect product $F_2 \ltimes F_3 \ltimes \ldots F_{n-1}$ of free groups.

16. *Let* $t_{i,j} := t_i t_{1+i} \ldots t_{j-1} \in S_n$ *in the permutation group. For every* $\pi \in S_n$ *there is a sequence* $\sigma_n(\pi) = (t_{i_1,n}, t_{i_2,n-1}, \ldots, t_{i_n,1})$*, the Schreier normal form of* π*, uniquely determined by the requirements that* $1 \le i_k \le 1+n-k$ *and that* $\pi = t_{i_1,n} t_{i_2,n-1} \ldots t_{i_n,1}$.

An n-permutation π can be achieved by first commuting the element $\pi^{-}(n)$ into its place (we are considering right-actions) by applying $t_{\pi^-(n),n}$. This leads to a problem in $n-1$ elements ♢

17. *For* $j \in \{1, \ldots, n\}$ *let* $U_j \le P_j \le B_n$ *be the subgroup of the non-permuting braid group* P_j *generated by the set* $\{\vartheta_{j,i}; i \in \{1, \ldots, j-1\}\}$*. For every braid* $\alpha \in B_n$ *there is a sequence* $(\tau_{i_1,n}, \ldots, \tau_{i_{n-1},2})$ *with* $i_j \in \{1, \ldots, n+1-j\}$ *and a sequence* $(\vartheta_{j_1}, \ldots, \vartheta_{j_{n-1}})$ *with* $\vartheta_{j_k} \in U_{1+k}$ *both uniquely determined by*

$$\alpha = \tau_{i_1,n} \ldots \tau_{i_{n-1},2} \vartheta_{j_1} \ldots \vartheta_{j_{n-1}}.$$

The map sending a braid α *to this pair of sequences is called Artin's combed normal form of the braid group.*

We follow [2], cor.1.8.2, pp.24. The set of products of the elements $\tau_{i_1,n}, \ldots, \tau_{i_{n-1},2}$ is a transversal (cross-section) of S_n in B_n with respect to P_n. This can be seen from the Schreier normal form for the permutation group S_n together with the homomorphism $\pi \in \mathrm{Hom}(B_n, S_n)$. It therefore exists such a unique product and a unique element $\vartheta \in P_n$ such that $\alpha = \tau_{i_1,n} \ldots \tau_{i_{n-1},2} \vartheta$. The structure of P_n as an $n-1$-fold

semidirect product of free groups, cf.thm.(11), yields a unique decomposition of the element ϑ. $\Diamond$

18. *Let $P_n^n = \{\alpha \in B_n; \pi(\alpha)(n) = n\} \leq B_n$ be the subgroup of the braid group that does not permute the n-th string (the n-pure braid group). Then P_n^n is isomorphic to the semidirect product $P_n^n \simeq B_{n-1} \ltimes U_n$, where B_{n-1} is imbedded into B_n by identifying the first $n-1$ strings and it acts onto U_n by conjugation.*

Let $\alpha = \tau_{i_1,n} \dots \tau_{i_{n-1},2}\vartheta$ be given in the combed normal form. If $\pi(\alpha)(n) = n$, then $i_1 = n$ s.t. $\tau_{i_1,n} = 1$. So α is an element of $B_{n-1}U_n$ and $P_n^n = B_{n-1}U_n$. U_n is normal in P_n^n and by the combed normal form, $B_{n-1} \cap U_n = \{1\}$, so the claim follows. For an algebraical proof of this decomposition, cf.[4] or [6], pp.153. $\Diamond$

In the following, F_n will be the free group generated by $\{f_1, \dots, f_n\}$. Subscripts may be dropped, if the rank of the group is clear or immaterial.

The explicit formula of the following imbedding theorem allows the application of section 2 to the braid group.

19 (E. Artin)**.** *Let $\iota \in \mathrm{Hom}(U_{1+n}, F_n)$, $\vartheta_{1+n,i} \mapsto f_i$ be an isomorphism of free groups. Let $\psi \in \mathrm{Hom}(B_n, \mathrm{Aut}(F_n))$ be such that $\psi(\alpha)(f) = \iota(\alpha^- \iota^{-1}(f)\alpha)$, $\alpha \in B_n$, $f \in F_n$. Then ψ is a monomorphism and*

$$\psi(\tau_i) : f_j \mapsto \begin{cases} f_i f_{i+1} f_i^{-1}, & j = i, \\ f_i, & j = i+1, \\ f_j, & j \notin \{i, i+1\} \end{cases} .$$

The existence of ψ as a homomorphism acting as claimed follows from the fact that $U_{1+n} \simeq F_n$ is invariant under conjugation with elements from B_n and from the relations between the elements τ_i and $\vartheta_{1+n,j}$. For a proof that the kernel of ψ is trivial, cf.[2], cor.1.8.3, pp.25. $\Diamond$

20. *The action of the elements $\vartheta_{n,i} \in B_n$ generating the free group U_n of rank $n-1$ in B_n onto the generators of the free group F_n is given by ($\epsilon \in \{-1, 1\}$)*

$$\vartheta_{n,i}^{\epsilon}(f_j) = \begin{cases} f_j, & j < i \\ \mathrm{Ad}((f_i f_n)^{\epsilon})(f_j), & j \in \{i, n\} \\ \mathrm{Ad}([f_n^{\epsilon}, f_i^{\epsilon}]^{-\epsilon})(f_j), & i < j < n \end{cases} ,$$

where $\mathrm{Ad}(x)(y) := xyx^-$, $[x, y] := xyx^-y^-$ for $x, y \in F_n$.

This can be inferred from equation (1) by imbedding $U_n \leq P_n \to P_{1+n}$ and $F_n \simeq U_{1+n} \leq P_{1+n}$. $\Diamond$

The procedure of taking a semidirect product $B_n \ltimes F_n$ and, by theorem (18), imbedding it into the braid group B_{1+n}, $B_n \ltimes F_n \simeq P_{1+n}^{1+n} \leq B_{1+n}$, can be iterated. We set

$$\begin{aligned} B_{n,m} &= B_n \ltimes F_n \ltimes \dots \ltimes F_{n+m-1}, \\ F_{n,m} &= F_n \ltimes \dots \ltimes F_{n+m-1}, \end{aligned}$$

such that $B_{n,m} = B_n \ltimes F_{n,m} \leq B_{n+m}$. In this iterated product, F_k is acting onto F_{1+k} according to thm.(20). $F_{n,m}$ may be identified with the subgroup of the pure braid group P_{n+m}, in which only the last m strings are not held fixed.

21. *The fundamental group of the space $Y_{x,m} = \{(y_1, \dots, y_m) \in X_m; y_i \neq x_j\}$ for $x \in X_n$ is isomorphic to the group $F_{n,m}$.*

The fiber bundle $Y_{n,m} \to X_{n+m} \to X_n$ due to vanishing of the higher homotopy groups yields the exact sequence $1 \to \pi_1(Y_{n,m}) \to \pi_1(X_{n+m}) \to \pi_1(X_n) \to 1$. We know, $\pi_1(X_{n+m}) \simeq P_{n+m}$ and $\pi_1(X_n) \simeq P_n$. The exact sequence $1 \to F_{n,m} \to P_{n+m} \to P_n \to 1$ can be derived from the combed normal form for P_{n+m}. By the five lemma, $F_{n,m} \simeq \pi_1(Y_{n,m})$. $\Diamond$

22. *The fiber $Y_{x,m} = \{(y_1, \dots, y_m) \in X_m; y_i \neq x_j\}$ over $x \in X_n$ has homology $H_*(Y_{x,m}, M) \simeq H_*(F_{n,m}, M)$ for any $F_{n,m}$ module M.*

We have shown, $Y_{x,m}$ is an Eilenberg-MacLane space of type $(F_{n,m}, 1)$. As a smooth manifold, it is homotopic to a cell complex, cf.[12], thm.3.5, pp.20 and p.36. So the claim follows e.g. from [3], par.III.1, pp.56, using Eilenberg's theorem in [15]. $\diamondsuit$

Now we are in the position to show, in which way the representations of B_n on $H_*(Y_{x,m}, \chi)$ derived in [8] can be constructed in purely algebraic terms.

4. Resolution of the integers over $F_{n,m}$

We construct a free $F_{n,m} \simeq \pi_1(Y_{x,m})$ resolution C of the integers. It is chosen in such a way that there is a particular braid action onto complexes $N \otimes_{F_{n,m}} C$ (and onto their homology) for any right $B_{n,m}$ module N. This action is described by the braid valued Burau matrices that will be derived in the next section.

For fixed integers n, m and for $k \in \{1, \dots, m\}$ let $F_{n+k-1} = F^k$ be the free group freely generated by the subset $\{f_i^{(k)}; i \in \{1, \dots, n+k-1\}\} \subset F^k$. Let I^k be the relative augmentation ideal of F^k in the group ring of $F_{n,k} = F^1 \ltimes \dots \ltimes F^k$. I^k is freely generated as a left module over $\{(f_i^{(k)} - 1); i \in \{1, \dots, n+k-1\}\}$. We distinguish I^k from its image in $\mathbb{Z}F_{n,k}$. Therefore we write s_i^k when we consider the generating elements in the ideal I^k rather than in $\mathbb{Z}F_{n,k}$. The natural imbedding $\iota : I^k \to \mathbb{Z}F_{n,k}$ thus sends $s_i^k \mapsto (f_i^{(k)} - 1)$. We consider the ideal I^k as an $F_{n,k}$ bimodule (and therefore as a bimodule over $F_{n,j} \leq F_{n,k}$, $j \leq k$). We sometimes write s^k for $(f^{(k)} - 1) \in I^k$, with $f^{(k)} \in F^k$.

23. *Let*

$$C_k^m := \oplus_i \mathbb{Z}F_{n,m} \otimes_{F_{n,i_1}} I^{i_1} \otimes_{F_{n,i_2}} \dots \otimes_{F_{n,i_k}} I^{i_k}$$

for $k \in \{1, \dots, m\}$, $1 \leq i_k < i_{k-1} < \dots < i_1 \leq m$ and let $C_0^m := \mathbb{Z}F_{n,m}$ with the natural structures as left $F_{n,m}$ modules. Let $\partial_k^m : C_k^m \to C_{k-1}^m$ be given by $\partial_k^m := \sum_{l=1}^k (-)^{1+l} \partial_{k,l}^m$, with

$$\begin{aligned} \partial_{k,l}^m : \mathbb{Z}F_{n,m} \otimes_{F_{n,i_1}} I^{i_1} \dots \otimes_{F_{n,i_{l-1}}} I^{i_{l-1}} \otimes_{F_{n,i_l}} I^{i_l} \otimes_{F_{n,i_{l+1}}} \dots I^{i_k} \\ \to \mathbb{Z}F_{n,m} \otimes_{F_{n,i_1}} I^{i_1} \dots \otimes_{F_{n,i_{l-1}}} I^{i_{l-1}} \iota(I^{i_l}) \otimes_{F_{n,i_{l+1}}} \dots I^{i_k}, \end{aligned}$$

where $\partial_{k,l}^m := 1 \otimes \dots \otimes \iota \otimes \dots \otimes 1$ and the imbedding $\iota : I^l \to \mathbb{Z}F_{n,l}$ occurs at the $(l+1)$-st position in the $(k+1)$-fold tensor product. Then

$$0 \to C_m^m \xrightarrow{\partial_m^m} C_{m-1}^m \xrightarrow{\partial_{m-1}^m} \dots \xrightarrow{\partial_1^m} C_0^m \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

with the augmentation $\epsilon : \mathbb{Z}F_{n,m} \to \mathbb{Z}$ sending $F_{n,m}$ to 1 is a complex of free left $F_{n,m}$ modules, augmented over the trivial left $F_{n,m}$ module $\mathbb{Z}$.

We will fix m and drop corresponding superscripts. Each C_k is a free left $F_{n,m}$ module, since all the factors in the defining tensor products are free over the respective rings. ∂_k is a homomorphism of left $F_{n,m}$ modules and Image(∂_1) is contained in the augmentation ideal of $F_{n,m}$ in $\mathbb{Z}F_{n,m}$, s.t. Image$(\partial_1) \leq$ Ker(ϵ). Furthermore for $k \in \{2, \dots, m\}$, Image$(\partial_k) \leq$ Ker(∂_{k-1}), i.e. $\partial_{k-1} \circ \partial_k = 0$, since in the sum expressing $\partial_{k-1} \circ \partial_k$, each tensor product with ι at the i-th and at the j-th position, $i \neq j$, occurs twice, once with sign $(-)^i(-)^{j-1}$ and once with sign $(-)^j(-)^i$. $\diamondsuit$

In explicit terms, the components of the boundary operator act as

$$\partial_{k,l}^m (s^{i_1} \otimes \dots s^{i_{l-1}} \otimes s^{i_l} \otimes \dots s^{i_k}) = s^{i_1} \otimes \dots s^{i_{l-1}} (f^{(i_l)} - 1) \otimes \dots s^{i_k},$$

where the product $s^{i_{l-1}}(f^{(i_l)} - 1)$ has to be understood as the right action of $I^{i_l} \leq \mathbb{Z}F_{n,i_l}$ onto the bimodule $I^{i_{l-1}}$. This expression for the boundary might be compared with the topological boundary matrix in [8], sec.3.2, p.152.

The complexes (C^m, ∂^m) can be described in recursive terms: for $k > 0$, $m > 0$, the modules are given by $C_0^m = \mathbb{Z}F_{n,m}$, $C_1^1 \simeq I^1$, $C_{1+k}^1 = \{0\}$,

$$C_k^{1+m} \simeq I^{1+m} \otimes_{F_{n,m}} C_{k-1}^m \oplus \mathbb{Z}F_{n,1+m} \otimes_{F_{n,m}} C_k^m.$$

The homomorphisms are determined by the augmentation $\epsilon^m : C_0^m \to \mathbb{Z}$, by $\partial_1^m : C_1^m \to C_0^m$, which imbeds the direct summands $I^k \to \mathbb{Z}F_{n,k}$ for all $k \in \{1, \ldots, m\}$ and by the recursive step

$$\begin{aligned}&\partial_k^{1+m}(j \otimes c + f \otimes d)\\ &= \quad (\partial_1^{1+m}(j) \otimes c - j \otimes \partial_{k-1}^m(c)) + f \otimes \partial_k^m(d),\end{aligned}$$

for $0 < k \le m$, $j \in I^{1+m}$, $c \in C_{k-1}^m$, $f \in F_{n,1+m}$, $d \in C_k^m$.

24. *The complex* $0 \to C_m^m \xrightarrow{\partial_m^m} C_{m-1}^m \xrightarrow{\partial_{m-1}^m} \ldots \xrightarrow{\partial_1^m} C_0^m \xrightarrow{\epsilon} \mathbb{Z} \to 0$ *of left* $F_{n,m}$ *modules is exact.*

Inductively for every $m > 0$ we will construct a free left $F_{n,m}$ resolution of the integers that coincides with the augmented complex C^m. Clearly, for $m = 1$ we have the resolution $0 \to I_{F_{n,1}} \to \mathbb{Z}F_{n,1} \to \mathbb{Z} \to 0$. Now we assume, there is an m s.t. $C^m \to \mathbb{Z} \to 0$ is a free resolution and the short sequences $0 \to K_i^m \to C_i^m \to K_{i-1}^m \to 0$, with $K_i^m = \mathrm{Ker}(\partial_i^m)$ are exact. In order to obtain the resolution C^{1+m}, in the first step we construct the diagram

$$\begin{array}{ccccc} I^{1+m} & \longrightarrow & K_0^{1+m} & \longrightarrow & K_0^m \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow & & \downarrow \\ I^{1+m} & \longrightarrow & C_0^{1+m} & \xrightarrow{p_0} & C_0^m . \\ {\scriptstyle 0}\downarrow & & {\scriptstyle \epsilon^{1+m}}\downarrow & & \downarrow{\scriptstyle \epsilon^m} \\ 0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} \end{array}$$

This will be shown to be commutative with exact rows and columns where the second maps in each row and and in each column are onto and the first maps are injective. The map $\mathbb{Z} \to \mathbb{Z}$ is the identity. The vertical map on the left, $I^{1+m} \to 0$, is obvious, the vertical map on the right, $\epsilon^m : C_0^m \to \mathbb{Z}$, is the augmentation. We set $C_0^{1+m} = \mathbb{Z}F_{n,1+m}$, which is a free left $F_{n,1+m}$ module and as an abelian group is isomorphic to the direct sum $I^{1+m} \oplus C_0^m$. The map $p_0 : C_0^{1+m} \to C_0^m$ is the surjection induced by the quotient map modulo the normal subgroup $F^{1+m} \lhd F_{n,1+m}$ and so it has the kernel I^{1+m}. The middle vertical map $\epsilon^{1+m} : C_0^{1+m} \to \mathbb{Z}$ is defined by commutativity of the two lower squares. It is the augmentation $\mathbb{Z}F_{n,1+m} \to \mathbb{Z}$. Therefore, the kernels of the vertical maps are I^{1+m}, $K_0^{1+m} = I_{F_{n,1+m}}$ and $K_0^m = I_{F_{n,m}}$, respectively. The maps in the first row are the induced ones and by the snake lemma, cf.[7], lemma III.5.1, p.99, this row is exact with the first map injective and the second one onto. Now, since I^{1+m} and $D_1 = \mathbb{Z}F_{n,1+m} \otimes_{F_{n,m}} C_1^m$ are free left $F_{n,1+m}$ modules, we obtain free presentations $0 \to I^{1+m} \to I^{1+m}$ and $L_1 \to D_1 \to K_0^m$ of the kernels, fitting into the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K_1^{1+m} & \xrightarrow{p_1} & L_1 \\ \downarrow & & \downarrow & & \downarrow \\ I^{1+m} & \longrightarrow & C_1^{1+m} & \xrightarrow{p_1} & D_1 . \\ {\scriptstyle \mathrm{Id}}\downarrow & & {\scriptstyle \partial_1^{1+m}}\downarrow & & \downarrow{\scriptstyle \Delta_1} \\ I^{1+m} & \longrightarrow & K_0^{1+m} & \xrightarrow{p_0} & K_0^m \end{array}$$

Again we will choose maps in such a way that exactness of all sequences is preserved, if we attach zero maps on both sides of the sequences and that the diagram is

commutative. The vertical map on the left, $I^{1+m} \to I^{1+m}$, is the identity, the vertical map on the right, $\Delta_1 : D_1 \to K_0^m$, is $\Delta_1 = \eta \otimes \partial_1^m$, $\eta : F_{n,1+m} \to F_{n,m}$ being the augmentation homomorphism with respect to F^{1+m}. $C_1^{1+m} = I^{1+m} \oplus D_1$ as free left $F_{n,1+m}$ modules and the middle row is induced by this decomposition. The free presentation $\partial_1^{1+m} : C_1^{1+m} \to K_0^{1+m}$ we are on the way of constructing as the middle column must make the lower squares commute. It is determined by its components. $\partial_{1,1}^{1+m} : I^{1+m} \to K_0^{1+m}$ is the imbedding $s_i^{1+m} \mapsto (f_i^{(1+m)} - 1)$ and $\partial_{1,2}^{1+m} : D_1 \to K_0^{1+m}$ maps $f \otimes_{F_{n,m}} s_i^k \mapsto f(f_i^{(k)} - 1) \in I_{F_{n,1+m}} = K_0^{1+m}$. Commutativity of the lower left square is obvious. To show $\Delta_1 = p_0 \circ \partial_{1,2}^{1+m}$, we notice, $p_0(\partial_{1,2}^m(fg \otimes s_i^k)) = f(f_i^{(k)} - 1)$ for $f \in F_{n,m}$, $g \in F^{1+m}$. On the other hand, $\Delta_1(fg \otimes s_i^k) = f(f_i^{(k)} - 1)$. Δ_1 is onto K_0^m, since ∂_1^m is, by assumption. So by the five lemma also ∂_1^{1+m} is onto. The kernel of the left vertical map is 0. The kernel of Δ_1 can be seen to be the module $L_1 = (I^{1+m} \otimes_{F_{n,m}} C_1^m) \oplus K_1^m$, using the decomposition $\mathbb{Z}F_{n,1+m} \simeq I^{1+m} \oplus \mathbb{Z}F_{n,m}$, where both direct sums are to be understood as direct sums of abelian groups, rather than $F_{n,1+m}$ modules. Again by the snake lemma, the row of kernels is exact, with a surjection after an injection. Once more we will repeat the construction of free presentations of the kernels occuring in the top row. This leads to an iterative procedure which eventually reaches an isomorphism s.t. the resolutions are finite, i.e. the vertical sequences begin with 0 at the top. Thus we consider the diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & K_2^{1+m} & \longrightarrow & L_2 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_2^{1+m} & = & D_2. \\
\downarrow & & \downarrow{\scriptstyle \partial_2^{1+m}} & & \downarrow{\scriptstyle \Delta_2} \\
0 & \longrightarrow & K_1^{1+m} & \xrightarrow{p_1} & L_1
\end{array}$$

Here $C_2^{1+m} = D_2 = I^{1+m} \otimes_{F_{n,m}} C_1^m \oplus \mathbb{Z}F_{n,1+m} \otimes_{F_{n,m}} C_2^m$. The map $D_2 \to L_1$ is $\Delta_2 = \partial_1^{1+m} \otimes 1 + 1 \otimes \partial_2^m$. It is onto, since by assumption ∂_2^m is onto K_1^m, and ∂_1^{1+m} is onto I^{1+m}. The free presentation $\partial_2^{1+m} : C_2^{1+m} \to K_1^{1+m}$ as a $F_{n,1+m}$ module map is uniquely determined by the requirement for commutativity of the lower right square, $p_1 \circ \partial_2^{1+m} = \Delta_2$. The components of ∂_2^{1+m} are $(\partial_1^{1+m} \otimes 1 - 1 \otimes \partial_1^m)$ and $1 \otimes \partial_2^m$ and it explicitly acts as

$$\begin{aligned}
&\partial_2^{1+m}(s_i^{1+m} \otimes_{F_{n,m}} c_1 \oplus f^{(1+m)} \otimes_{F_{n,m}} c_2) \\
&= \quad (f_i^{(1+m)} - 1) \otimes c_1 - s_i^{1+m} \partial_1^m(c_1) + f_i^{(1+m)} \otimes_{F_{n,m}} \partial_2^m(c_2),
\end{aligned}$$

for $c_1 \in C_1^m$, $c_2 \in C_2^m$, $f^{(1+m)} \in F_{n,1+m}$. The map ∂_2^{1+m} is onto, since Δ_2 is. The kernel of the map Δ_2 is the module $L_2 = \mathbb{Z}F_{n,1+m} \otimes K_2^m + < s_i^{1+m} \otimes \partial_2^m(c) - (f_i^{(1+m)} - 1) \otimes c >$. To validate this, one uses the decomposition $\mathbb{Z}F_{n,1+m} \simeq I^{1+m} \oplus \mathbb{Z}F_{n,m}$ and the linear independence of $\{f_i^{(1+m)} - 1\}$. The snake lemma again shows that the top row is exact with a surjection after an injective map. Therefore $\mathrm{Ker}(\partial_2^{1+m}) = K_2^{1+m} = L_2$. Proceeding iteratively in this way, using the induction assumption we find, $C_{1+l}^{1+m} = I^{1+m} \otimes_{F_{n,m}} C_l^m \oplus \mathbb{Z}F_{n,1+m} \otimes_{F_{n,m}} C_{1+l}^m$, $K_{1+l}^{1+m} = \mathbb{Z}F_{n,1+m} \otimes K_l^m + < s_i^{1+m} \otimes \partial_l^m(c) - (f_i^{(1+m)} - 1) \otimes c >$. $\Diamond$

For every right $B_{n,m}$ module N, the tensor complex $N \otimes_{F_{n,m}} C$ and its homology $H_*(F_{n,m}, N)$ have a structure as a right B_n module. The right action is given by $\alpha : n \otimes_{F_{n,m}} c \mapsto n\alpha \otimes_{F_{n,m}} \alpha(c)$, where $\alpha \in B_n$ acts via Artin's automorphisms onto $c \in C^k$. We prefer a different point of view. Instead of the augmentation ideals $I^l = I_{F_{n+l-1}, F_{n,l}}$ used in the construction of the resolution C, we may consider the

ideals $\bar{I}^l = I_{F_{n+l-1},B_{n,l}}$, with $B_{n,l} = B_n \ltimes F_{n,l}$. $\bar{I}^l$ as a left $B_{n,l}$ module is isomorphic to $\mathbb{Z}B_{n,l} \otimes_{F_{n,l}} I^l$ and as an ideal in the ring $\mathbb{Z}B_{n,l}$, it carries a structure as right $B_{n,l}$ module, too. Let the complex $\bar{C}$ be constructed similar to C in thm. (23) by setting

$$\bar{C}_k^m := \oplus_i \mathbb{Z}B_{n,m} \otimes_{B_{n,i_1}} \bar{I}^{i_1} \otimes_{B_{n,i_2}} \dots \otimes_{B_{n,i_k}} \bar{I}^{i_k}.$$

This complex is not a free resolution of $\mathbb{Z}$ anymore but it is still exact and for any right $B_{n,m}$ module N we may identify $N \otimes_{F_{n,m}} C \simeq N \otimes_{B_{n,m}} \bar{C}$ as right B_n modules, where the B_n action on $N \otimes_{B_{n,m}} \bar{C}$ is more obvious. This action is known as soon as the representations on the ideals I_{F_k,B_kF_k} for all k are known, because $\bar{I}^l = I_{F_{n+l-1},B_{n,l}}$ is imbedded into I_{F_k,B_kF_k} with $k = n+l-1$. This action will be found in the next section.

5. The braid valued Burau matrices

The relative augmentation ideal I_{F_n,B_nF_n} is a right B_n module and is free as a left B_nF_n module. The right B_n action therefore is determined by matrices which also determine the action onto the complex $N \otimes_{F_{n,m}} C$ and onto its homology, for any right $B_{n,m}$ module N. They generalise the Burau matrices.

The Burau representation of the braid group is a “classical” Birman-Magnus representation in the sense of section 2, cf.[2, 11]. Let $\psi \in \mathrm{Hom}(F_n, F_1 =< t >)$. This map has the kernel $U = \{f_{i_1}^{\epsilon_1} \dots f_{i_k}^{\epsilon_k}; \sum \epsilon_l = 0\}$. According to thm. (19), for every braid $\alpha \in B_n$ and every $f \in F_n$, $\psi(\alpha(f)) = \psi(f)$. So there is a Birman module $M_\psi \simeq \mathbb{Z}[t] \otimes_{B_nF_n} I_{F_n,B_nF_n}$ carrying a right action of B_n,

$$\begin{array}{rcl} s_j\tau_i & = & \delta_\psi(\tau_i(f_j)) \\ & = & \left\{ \begin{array}{rcll} \delta_\psi(f_if_{i+1}f_i^{-1}) & = & (1-t)s_i + ts_{i+1}, & j = i \\ \delta_\psi(f_i) & = & s_i, & j = i+1 \\ \delta_\psi(f_j) & = & s_j, & j \notin \{i, i+1\} \end{array} \right. . \end{array}$$

This means the right action is determined by the well-known Burau matrices

$$\tau_i \mapsto \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 & 0 \\ 0 & (1-t) & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-i-1} \end{pmatrix}.$$

The Magnus submodule $\delta_\psi(U) \simeq U/[U,U] \simeq H_1(F_n, \mathbb{Z}[t])$ according to thm. (5) is

$$\{\sum g_js_j; g_j \in \mathbb{Z}[t], \sum g_j(\psi(f_j) - 1) = 0\} = \{\sum g_js_j; g_j \in \mathbb{Z}[t], \sum g_j = 0\},$$

since $\psi(f_j) = t$ and the polynomial ring $\mathbb{Z}[t]$ is free of zero divisors.

The modules constructed in section 2 according to thms. (6) and (7) are induced by the representation on the relative augmentation ideal $I_{F,EF}$. In the last section we have seen that the right B_n action onto $N \otimes_{F_{n,m}} C$ for any right $B_{n,m}$ module also is determined by the ideals $I_{F,BF}$. So we obtain the braid valued Burau representation, [5, 10].

25. *Let M be a free left BF module of rank n with basis $\{s_1, \dots, s_n\}$. Then the map $\rho : \{\tau_1, \dots, \tau_{n-1}\} \to \mathrm{End}_{BF}(M)$ defined by*

$$\rho(\tau_i)(s_j) = \tau_i \left\{ \begin{array}{ll} (1 - f_if_{i+1}f_i^{-1})s_i + f_is_{i+1}, & j = i \\ s_i, & j = i+1 \\ s_j, & j \notin \{i, i+1\} \end{array} \right. ,$$

uniquely extends to a monomorphism $\rho \in \mathrm{Hom}(B, \mathrm{End}_{BF} M)$.

$I_{F,BF}$ is a free left BF module over the set $\{(f_i - 1); i \in \{1, \dots, n\}\}$, so we identify M with $I_{F,BF}$ via $s_i \mapsto (f_i - 1)$. By multiplication from the right $I_{F,BF}$

is a module over B, $(f_j - 1) \mapsto (f_j - 1)\tau_i = \tau_i(\tau_i(f_j) - 1)$. The element $\tau_i(f_j)$ is determined by Artin's thm. (19) so we obtain the equation as claimed. $\diamondsuit$

In terms of matrices over the ring $\mathbb{Z}(BF)$ the representation of τ_i is given by

$$\tau_i \mapsto \tau_i \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 & 0 \\ 0 & (1 - f_i f_{i+1} f_i^{-1}) & f_i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-i-1} \end{pmatrix}. \tag{3}$$

With the help of the imbedding $B_n F_n \to B_{1+n}$, $\tau_i f_j \mapsto \tau_i \vartheta_{1+n,j}$, the semidirect product $B_n F_n$ and therefore the matrix elements of eqn. (3) can again be represented by the braid valued Burau matrices. By m-fold iteration we obtain representations of B_n (or of subgroups $S \leq B_n$) in terms of matrices with values in the ring $\mathbb{Z}(B_n F_n F_{1+n} \dots F_{n+m-1})$ ($\mathbb{Z}(S F_n F_{1+n} \dots F_{n+m-1})$, respectively).

Finally we show that the local coefficient system chosen in [8] enables us to identify the recursion matrices given there with images of the braid valued Burau matrices. We imbed $F_{n,m} = F_n \ltimes \dots F_{n+m-1}$ into B_{n+m}. The generators $\{f_i^{(k)}; i \in \{1, \dots, n+k-1\}\}$ of F_{n+k-1} are mapped as $f_i^{(k)} \mapsto \vartheta_{k+n,i}$.

26. *Let a homomorphism $\chi \in Hom(F_{n,m}, \mathbb{C}\backslash\{0\})$ be given by the map $\vartheta_{l,i} \mapsto q_{l,i}$, for $l \in \{1+n, \dots, m+n-1\}$, $i < l$. If $q_{l,i} = q_{l,j}$ for $i, j \in \{1, \dots, n\}$, then χ extends to a map $\bar{\chi} \in Hom(B_n \ltimes F_{n,m}, \mathbb{C}\backslash\{0\})$ by setting $\bar{\chi}(\alpha\vartheta) = \chi(\vartheta)$, $\alpha \in B_n$, $\vartheta \in F_{n,m}$.*

Eqn. (1) shows that any map $\vartheta_{l,i} \mapsto q_{l,i}$ of the generators into an abelian group extends to a homomorphism of $F_{n,m}$. This is also the content of lemma 2.1, p.145, [8]. The relations defining the semidirect product $B_{n,m} = B_n \ltimes F_{n,m}$ are those in B_n, in $F_{n,m}$ and in addition those between B_n and $F_{n,m}$:

$$\tau_i^{-1} \vartheta_{l,j} \tau_i = \begin{cases} \vartheta_{l,i} \vartheta_{l,i+1} \vartheta_{l,i}^{-1}, & j = i \\ \vartheta_{l,i}, & j = i+1 \\ \vartheta_{l,j}, & j \notin \{i, i+1\} \end{cases}.$$

So the conditions $q_{l,i} = q_{l,j}$ allow the extension of χ to $B_{n,m}$ by mapping B_n to 1. $\diamondsuit$

From the following matrices, representations of B_n on the homology $H_m(Y_{x,m}, \chi)$ are derived in [8].

27 (R.J.Lawrence)**.** *For $r \in \{0, 1, \dots, m\}$, $l \in \{1+r, \dots, m+n-1\}$, let the matrices $A_l^{(r)}$ be recursively defined by the equations*

$$\begin{aligned} A_i^{(0)} &= 1, \\ A_i^{(k)} &= \begin{pmatrix} \mathbf{1}_{m+n-k-i} & 0 & 0 & 0 \\ 0 & 0 & b_{k,1+i}^{(k-1)} & 0 \\ 0 & 1 & (1 - (b_{k,1+i}^{(k-1)})^{-1} b_{k,i}^{(k-1)} b_{k,1+i}^{(k-1)}) & 0 \\ 0 & 0 & 0 & \mathbf{1}_{i-2} \end{pmatrix} A_i^{(k-1)}, \end{aligned}$$

where for $q_{1+r,p} \in \mathbb{C}\backslash\{0\}$, $p > 1+r$ we set

$$\begin{aligned} b_{1+r,p}^{(r)} &= A_{1+r,p}^{(r)2} / q_{1+r,p}, \\ A_{1+r,p}^{(r)} &= A_{1+r}^{(r)} A_{2+r}^{(r)} \dots A_{p-1}^{(r)} A_{p-2}^{(r)-} \dots A_{1+r}^{(r)-}. \end{aligned}$$

If $q_{j,s} = q_{j,t}$ for $j \in \{1, \dots, m\}$, $s, t \in \{1+m, \dots, n+m\}$, then the matrices $A_l^{(m)}$, $l \in \{1+m, \dots, m+n-1\}$ generate a homomorphic image of the braid group B_n.

This is thm. 3.4, pp.156, [8], where it is proved by computing the braid action on a cell complex. We have only given the recursion matrices $A_l^{(r)} = A_{l,1+l}^{(r)}$ instead of the more complicated ones $A_{l,p}^{(r)}$, because the matrices $A_l^{(r)}$ generate the others.

We have furthermore used lemma 3.3, p.153, parts (i) and (ii) to slightly change the form of $A_l^{(r)}$ compared to the expression in loc. cit. $\diamondsuit$

The theorem can be understood in our algebraic approach. We compare $A_l^{(r)}$ to the braid valued Burau matrix, eqn. (3). Lawrence's matrices act by left multiplication with column vectors, the generalised Burau matrices act by right multiplication with row vectors. Furthermore, the conventions in [8] on the imbedding $B_{n,m} \to B_{n+m}$ are different from ours. This accounts for the reversal of products of matrices, for rearranging of rows and columns and for reindexing of the generators. Apart from this both matrices have the same appearance. The lowest order matrices due to lemma (26) yield a representation $\bar{\chi} \in \mathrm{Hom}(B_{n,m}, \mathbb{C}\backslash\{0\})$, $\tau_{m+i} \to A^0_{m+i} = 1$, $\vartheta_{l,p} \mapsto q^-_{l,p}$, $i \in \{1, \dots, n-1\}$, $l \in \{1, \dots, m\}$, $l < p$ (where now we use the imbedding of [8]). By the braid valued Burau matrices and the similarity of the recursion matrix with these, the matrices $A_l^{(m)}$ for $l \in \{1+n, \dots, n+m-1\}$ represent B_n, if the matrices $A_l^{(m-1)}$, $l \in \{1+m, \dots, n+m-1\}$ and $b^{(m-1)}_{m,p}$, $p \in \{1+m, \dots, n+m-1\}$ represent $B_n F_n$. So finally we are led to a representation of $B_n F_n \dots F_{n+m-1}$ which is given by $\bar{\chi}$.

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